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# On the existence of a Morley sequence with a certain property

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## Abstract

Simplicity (in model theoretic sense) was originally defined by Shelah in [0]. Under simplicity, he showed the existence of a Morley sequence starting from a given element. Kim [1] showed various important results on simple theories. The existence of a Morley sequence played a key role in his proofs. In this note, we prove a result on the existence of a Morley sequence of a certain form.

$T$  is a simple theory formulated in the language  $L$ . We work in an  $\omega$ -saturated model  $M$  of  $T$ . Elements of  $M$  will be denoted by  $a, b, a_i, a_{ij}, \dots$  and so on. Finite tuples of elements of  $M$  will be denoted by  $\bar{a}, \bar{a}_i, \dots$ . Subsets of  $M$  will be denoted by  $A, B, \dots$ . A sequence  $\{\bar{a}_i\}_{i \in \omega}$  is called an indiscernible sequence over  $A$  if whenever  $\{i_1 < \dots < i_n\}$  and  $\{j_1 < \dots < j_n\}$  are finite subsets of  $\omega$ , and  $\varphi(x_1, \dots, x_n)$  is an  $L(A)$ -formula then

$$\varphi(a_{i_1}, \dots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \dots, a_{j_n})$$

holds in  $M$ . An indiscernible sequence  $\{\bar{a}_i\}_{i \in \omega}$  over  $A$  is called a Morley sequence over  $A$  if the sequence is an independent set over  $A$ . (See [0] and [1].)

In Kolesnikov's attempt ([2]) for dividing simple theories into countably many classes, he showed that for any finite independent set  $\{a_1, \dots, a_n\}$ , there is a Morley sequence  $\{\bar{b}_i : i \in \omega\}$  such that  $b_{ii} = a_i$  for  $i = 1, \dots, n$ , where  $b_{ij}$  is the  $j$ -th element of  $\bar{b}_i$ . In this short note, we extend his result slightly.

**Proposition 1** *Let  $T$  be simple and  $\{a_0, a_1, \dots, a_{n-1}\}$  an independent set. For  $i < n$ , let  $I_i$  be a Morley sequence starting from  $a_i$ . Then there is an independent set  $A = \{a_{ij} : i < n, j \in \omega\}$  with  $a_{00} = a_0, \dots, a_{0,n-1} = a_n$  such that for each  $i < n$ ,*

1.  $\{a_{ij} : j \in \omega\}$  (the  $i$ -th row of the matrix  $A$ ) is isomorphic to  $I_i$ ;
2.  $\{\bar{a}_{ij} : j \in \omega\}$  is a Morley sequence over  $a_{i+1}, \dots, a_{n-1}$ , where  $\bar{a}_{ij} = a_{0j}, \dots, a_{ij}$ .

*Proof:* By induction on  $i < n$ , we shall choose  $J_i = \{a_{ij} : j \in \omega\}$  satisfying the following conditions.

- $J_0 \cup \dots \cup J_i \cup \{a_{i+1}, \dots, a_n\}$  is an independent set;
- $a_{00} = a_0, \dots, a_{i0} = a_i$ ;
- $J_i \cong I_i$ .
- $\{\bar{a}_{ij} : j \in \omega\}$  is a Morley sequence over  $a_{i+1}, \dots, a_{n-1}$ , where  $\bar{a}_{ij} = a_{0j}, \dots, a_{ij}$ .

Suppose that we have chosen  $J_j$ 's for  $j < i$ . So  $K = \{\bar{a}_{i-1,j} : j \in \omega\}$  is a Morley sequence over  $a_i, a_{i+1}, \dots, a_{n-1}$ . We can choose a copy  $J_i = \{a_{ij} : j \in \omega\}$  of  $I_i$  over  $a_i$  such that  $J_i$  is a Morley sequence over  $a_{i+1}, \dots, a_{n-1}$ . Let us consider the type  $q_{a_i, a_{i+1}, \dots, a_n} = \text{tp}(K/a_i, a_{i+1}, \dots, a_n)$ . By our condition,  $K$  and  $a_i, a_{i+1}, \dots, a_{n-1}$  are independent. So the type  $\bigcup_{b \in I_i} q_{b, a_{i+1}, \dots, a_n}$  is consistent, and does not fork over  $\emptyset$ . So there is a copy  $K'$  of  $K$  realizing  $\bigcup_{b \in I_i} q_{b, a_{i+1}, \dots, a_n}$  such that  $K'$  and  $J_i \cup \{a_{i+1}, \dots, a_{n-1}\}$  are independent. Using an  $a_i, \dots, a_n$ -isomorphism sending  $K'$  to  $K$ , we may assume that  $K$  and  $J_i \cup \{a_{i+1}, \dots, a_n\}$  are independent from the first. Now, by compactness argument using Ramsey's theorem, we may further assume that  $\{\bar{a}_{i,j} : j \in \omega\}$  is an indiscernible sequence over  $a_{i+1}, \dots, a_n$ , so it is a Morley sequence over  $a_{i+1}, \dots, a_n$ .

**Corollary 2** *Let  $T$  be simple and  $\{a_0, a_1, \dots, a_{n-1}\}$  an independent set. Then there is an independent set  $A = \{a_{ij} : i < n, j \in \omega\}$  satisfying the following properties:*

1. *The columns of  $A$  form a Morley sequence, in other words, if we put  $\bar{a}_i = a_{0i} \dots a_{n-1,i}$ , then  $I = \{\bar{a}_i : i \in \omega\}$  is a Morley sequence.*
2. *For any  $i_0, \dots, i_{n-1} \in \omega$ ,  $\text{tp}(a_{0,i_0}, \dots, a_{n-1,i_{n-1}}) = \text{tp}(a_0, \dots, a_{n-1})$ .*

*Proof:* Choose any  $A = \{a_{ij} : i < n, j \in \omega\}$  satisfying conditions stated in the proposition above. It is sufficient to prove the second item. But it can be shown by the iterated use of the second item in the proposition.

**References.**

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